The mathematical derivation of the Franck–Condon principle was sloppy in the lecture (sorry). The interaction energy between the oscillating x-polarized electric field and electrical charges in a molecule consists of two terms,

$$\hat{H}^{(1)} = E\cos(\omega t)\left(\sum_i -e\hat{x}_i + \sum_I Z_I e\hat{x}_I\right), \tag{1}$$

where E is the field amplitude, ω is the frequency of light, $-e$ is the electric charge of electron, x_i is the x coordinate of the ith electron, Z_I is the atomic number of the Ith nucleus ($Z_I e$ is its electric charge), and x_I is the x coordinate of the Ith nucleus.

The transition dipole moment (the integral in the equation below), which is derivable from the first-order time-dependent perturbation theory with the above perturbation operator, therefore, has two terms, too:

$$I_{m\leftarrow n} \propto E^2\left|\int \Psi_m^*\left(\sum_i -e\hat{x}_i + \sum_I Z_I e\hat{x}_I\right)\Psi_n\,d\tau\right|^2, \tag{2}$$

which is why it can be divided into two separate integrals:

$$\begin{aligned}
&\int \Psi_m^*\left(\sum_i -e\hat{x}_i + \sum_I Z_I e\hat{x}_I\right)\Psi_n\,d\tau \\
&= \int \psi_m^{e*}\psi_m^{v*}\left(\sum_i -e\hat{x}_i + \sum_I Z_I e\hat{x}_I\right)\psi_n^e\psi_n^v\,d\tau_e d\tau_v \\
&= \int \psi_m^{e*}\left(\sum_i -e\hat{x}_i\right)\psi_n^e\,d\tau_e \int \psi_m^{v*}\psi_n^v\,d\tau_v + \int \psi_m^{e*}\psi_n^e\,d\tau_e \int \psi_m^{v*}\left(\sum_I Z_I e\hat{x}_I\right)\psi_n^v\,d\tau_v,
\end{aligned} \tag{3}$$

where the Born–Oppenheimer separation of the electronic (‘e’) and nuclear (‘v’) degrees of freedom has been made. Operators $\hat{x}_i$ act only on the electronic wave functions ψ^e and integrated over electronic coordinates τ_e, while operators $\hat{x}_I$ act on the nuclear wave functions ψ^v and integrated over nuclear coordinates τ_v.

In the lecture, the two components of the operator were collectively referred to as ‘$\hat{x}$’, which was incorrect and did not explain why the integral splits into two terms.